Matrix Concepts -- Explanations
This module introduces basic properties of matrices: concepts, multiplication by a constant, addition and subtraction, and setting two matrices equal to one another.

Conceptual Explanations: Matrices

A "matrix" is a grid, or table, of numbers. For instance, the following matrix represents the prices at the store "Nuthin' But Bed Stuff."

	King-sized	Queen-sized	Twin
Mattress	$649	$579	$500
Box spring	$350	$250	$200
Fitted sheet	$15	$12	$10
Top sheet	$15	$12	$10
Blanket	$20	$20	$15

(The matrix is the numbers, not the words that label them.)

Of course, these prices could be displayed in a simple list: "King-sized mattress," "Queen-sized mattress," and so on. However, this two-dimensional display makes it much easier to compare the prices of mattresses to box springs, or the prices of king-sized items to queen-sized items, for instance.

Each horizontal list of numbers is referred to as a **row;** each vertical list is a **column.** Hence, the list of all mattresses is a row; the list of all king-sized

prices is a column. (It's easy to remember which is which if you think of Greek columns, which are big posts that hold up buildings and are very tall and...well, you know...vertical.) This particular matrix has 5 rows and 3 columns. It is therefore referred to as a 5×3 (read, "5 by 3") matrix.

If a matrix has the same number of columns as rows, it is referred to as a **square matrix**.

Adding and Subtracting Matrices

Adding matrices is very simple. You just add each number in the first matrix, to the corresponding number in the second matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} = \begin{bmatrix} 61 & 52 & 43 \\ 34 & 25 & 16 \end{bmatrix}$$

For instance, for the upper-right-hand corner, the calculation was $3 + 40 = 43$. Note that both matrices being added are 2×3, and the resulting matrix is also 2×3. **You cannot add two matrices that have different dimensions.**

As you might guess, subtracting works much the same way, except that you subtract instead of adding.

$$\begin{bmatrix} 60 & 50 & 40 \\ 30 & 20 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 59 & 48 & 37 \\ 26 & 15 & 4 \end{bmatrix}$$

Once again, note that the resulting matrix has the same dimensions as the originals, and that **you cannot subtract two matrices that have different dimensions.**

Multiplying a Matrix by a Constant

What does it mean to multiply a number by 3? It means you add the number to itself 3 times.

Multiplying a **matrix** by 3 means the same thing...you add the matrix to itself 3 times.

$$3\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{bmatrix}3 & 6 & 9\\ 12 & 15 & 18\end{bmatrix}$$

Note what has happened: each element in the original matrix has been multiplied by 3. Hence, we arrive at the method for multiplying a matrix by a constant: you multiply each element by that constant. The resulting matrix has the same dimensions as the original.

$$\frac{1}{2}\begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{bmatrix} = \begin{matrix}\frac{1}{2} & 1 & \frac{3}{2}\\ 2 & \frac{5}{2} & 3\end{matrix}$$

Matrix Equality

For two matrices to be “equal” they must be exactly the same. That is, they must have the same dimensions, and each element in the first matrix must be equal to the corresponding element in the second matrix.

For instance, consider the following matrix equation.

$$\begin{bmatrix}1 & + \\ 12 & 10\end{bmatrix} = \begin{bmatrix}1 & 18\\ - & 10\end{bmatrix}$$

Both matrices have the same dimensions. And the upper-left and lower-right elements are definitely the same.

But for the matrix to be equal, we also need the other two elements to be the same. So

$$+ \quad = 18$$

$$- \quad = 12$$

Solving these two equations (for instance, by elimination) we find that $= 15, \quad = 3.$

You may notice an analogy here to complex numbers. When we assert that two complex numbers equal each other, we are actually making two statements: the **real** parts are equal, and the **imaginary** parts are equal. In such a case, we can use one equation to solve for two unknowns. A very similar situation exists with matrices, except that one equation actually represents **many more** statements. For 2×2 matrices, setting them equal makes four separate statements; for 2×3 matrices, six separate statements; and so on.

OK, take a deep breath. Even if you've never seen a matrix before, the concept is not too difficult, and everything we've seen so far should be pretty simple, if not downright obvious.

Let that breath out now. This is where it starts to get weird.

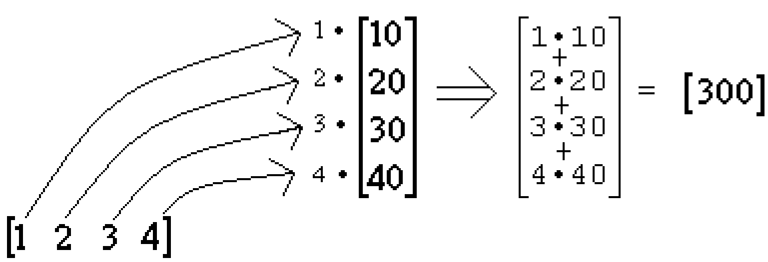
Matrix Concepts -- Multiplying Matrices
This module covers multiplication of matrices.

Multiplying a Row Matrix by a Column Matrix

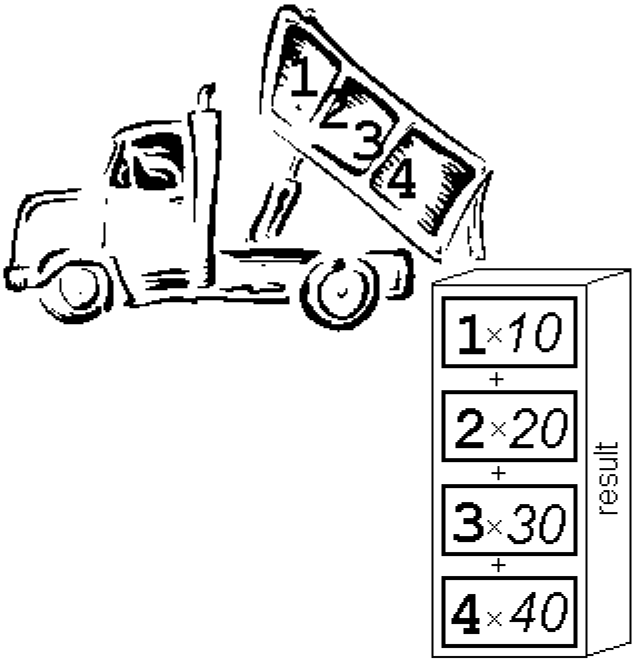
A “row matrix” means a matrix with only one row. A “column matrix” means a matrix with only one column. When a row matrix has the same number of elements as a column matrix, they can be multiplied. So the following is a perfectly **legal** matrix multiplication problem:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \text{ x } \begin{matrix} 10 \\ 20 \\ 30 \\ 40 \end{matrix}$$

These two matrices could not be added, of course, since their dimensions are different, but they **can** be multiplied. Here’s how you do it. You multiply the first (left-most) item in the row, by the first (top) item in the column. Then you do the same for the second items, and the third items, and so on. Finally, you **add** all these products to produce the final number.



A couple of my students (Nakisa Asefnia and Laura Parks) came up with an ingenious trick for visualizing this process. Think of the row as a dump truck, backing up to the column dumpster. When the row dumps its load, the numbers line up with the corresponding numbers in the column, like so:



So, without the trucks and dumpsters, we express the result—a row matrix, times a column matrix—like this:

$$\begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \quad \begin{matrix} 10 \\ 20 \\ 30 \\ 40 \end{matrix} = 300$$

There are several subtleties to note about this operation.

- The picture is a bit deceptive, because it might appear that you are multiplying two columns. In fact, **you cannot multiply a column matrix by a column matrix**. We are multiplying a row matrix by a column matrix. The picture of the **row** matrix “dumping down” only demonstrates which numbers to multiply.
- The answer to this problem is not a number: it is a 1-by-1 matrix.
- The multiplication can only be performed if the **number of elements** in each matrix is the same. (In this example, each matrix has 4 elements.)
- Order matters! We are multiplying a **row matrix times a column matrix**, not the other way around.

It’s important to practice a few of these, and get the hang of it, before you move on.

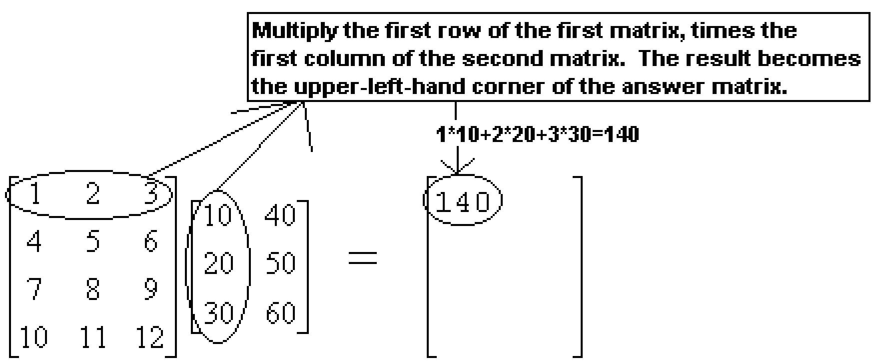
Multiplying Matrices in General

The general algorithm for multiplying matrices is built on the row-times-column operation discussed above. Consider the following example:

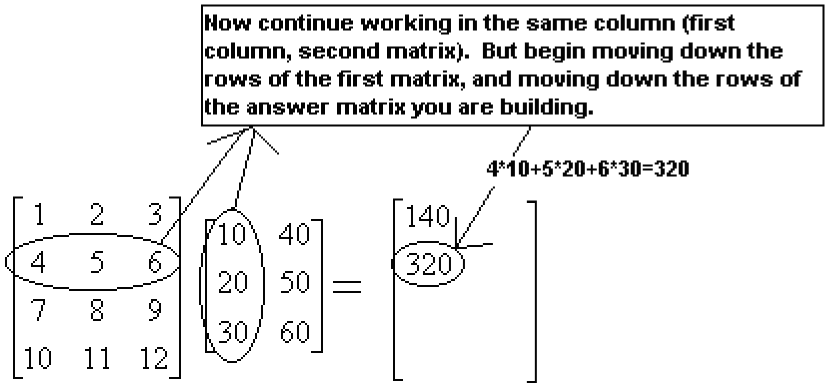
$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix} \qquad \begin{matrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{matrix}$$

The key to such a problem is to think of the first matrix as a list of **rows** (in this case, 4 rows), and the second matrix as a list of **columns** (in this case, 2 columns). You are going to multiply each **row** in the first matrix, by each **column** in the second matrix. In each case, you will use the “dump truck” method illustrated above.

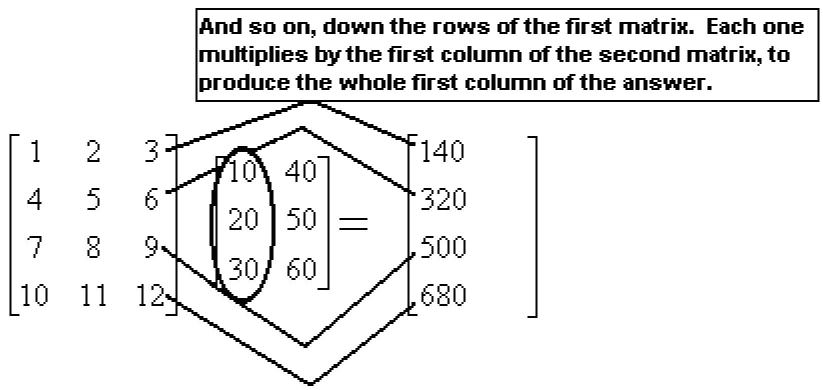
Start at the beginning: first row, times first column.



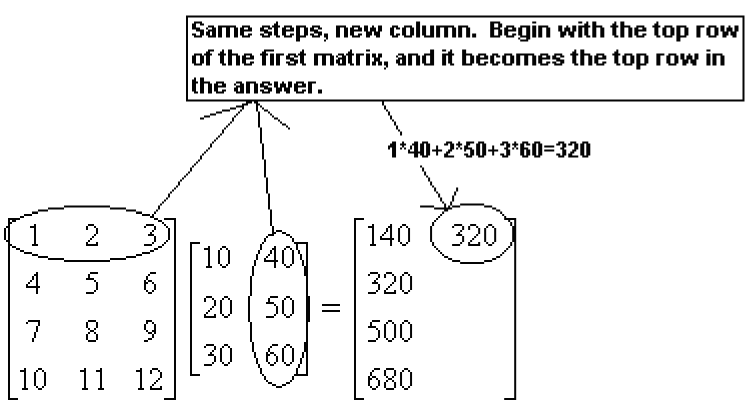
Now, move **down** to the next row. As you do so, move **down** in the answer matrix as well.



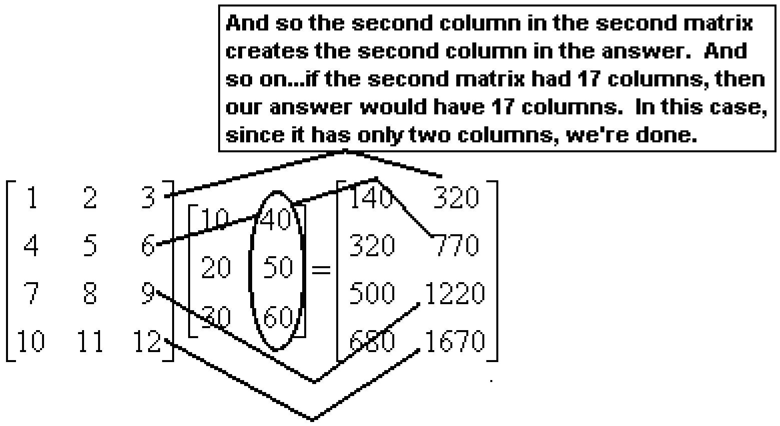
Now, move down the rows in the first matrix, multiplying each one by that same column on the right. List the numbers below each other.



The **first column** of the second matrix has become the **first column** of the answer. We now move on to the **second column** and repeat the entire process, starting with the first row.



And so on, working our way once again through all the rows in the first matrix.



We’re done. We can summarize the results of this entire operation as follows:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} \begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} = \begin{bmatrix} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{bmatrix}$$

It’s a strange and ugly process—but everything we’re going to do in the rest of this unit builds on this, so it’s vital to be comfortable with this process. **The only way to become comfortable with this process is to do it. A lot.** Multiply a lot of matrices until you are confident in the steps.

Note that we could add more rows to the first matrix, and that would add more rows to the answer. We could add more **columns** to the second matrix, and that would add more columns to the answer. However—if we added a column to the first matrix, or added a row to the second matrix, we would have an illegal multiplication. As an example, consider what happens if we try to do this multiplication in reverse:

$$\begin{bmatrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$ **Illegal multiplication**

If we attempt to multiply these two matrices, we start (as always) with the first row of the first matrix, times the first column of the second matrix:

$$\begin{bmatrix} 10 & 40 \end{bmatrix} \begin{matrix} 1 \\ 4 \\ 7 \\ 10 \end{matrix}$$

. But this is an illegal multiplication; the items don't line up, since there are two elements in the row and four in the column. So you cannot multiply these two matrices.

This example illustrates two vital properties of matrix multiplication.

- The number of columns in the first matrix, and the number of rows in the second matrix, must be equal. Otherwise, you cannot perform the multiplication.
- Matrix multiplication is not **commutative**—which is a fancy way of saying, order matters. If you reverse the order of a matrix multiplication, you may get a different answer, or you may (as in this case) get no answer at all.

Matrix Concepts -- The Identity Matrix
This module introduces the identity matrix and its properties.

When multiplying numbers, the number 1 has a special property: when you multiply 1 by any number, you get that same number back. We can express this property as an algebraic generalization:
Equation:

$$1x = x$$

The matrix that has this property is referred to as the **identity matrix**.

Definition of Identity Matrix
The **identity matrix**, designated as $[I]$, is defined by the property:
$[A][I] = [I][A] = [A]$

Note that the definition of [I] stipulates that the multiplication must **commute**—that is, it must yield the same answer no matter which order you multiply in. This is important because, for most matrices, multiplication does not commute.

What matrix has this property? Your first guess might be a matrix full of 1s, but that doesn't work:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 7 & 7 \end{bmatrix}$	**so** $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **is not an identity matrix**

The matrix that **does** work is a diagonal stretch of 1s, with all other elements being 0.

$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$	**so** $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ **is the identity for 2x2 matrices**
$\begin{matrix}2 & 5 & 9\\ \pi & -2 & 8\\ -3 & 1/2 & 8.3\end{matrix}\;\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}\;=$ $\begin{matrix}2 & 5 & 9\\ \pi & -2 & 8\\ -3 & 1/2 & 8.3\end{matrix}$	$\begin{matrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{matrix}$ **is the identity for 3x3 matrices**

You should confirm those multiplications for yourself, and also confirm that they work in reverse order (as the definition requires).

Hence, we are led from the definition to:

The Identity Matrix
For any square matrix, its identity matrix is a diagonal stretch of 1s going from the upper-left-hand corner to the lower-right, with all other elements being 0. **Non-square** matrices do not have an identity. That is, for a non-square matrix $[A]$, there is no matrix such that $[A][I]=[I][A]=[A]$.

Why no identity for a non-square matrix? Because of the requirement of commutativity. For a non-square matrix $[A]$ you might be able to find a matrix $[I]$ such that $[A][I]=[A]$; however, if you reverse the order, you will be left with an illegal multiplication.

Matrix Concepts -- The Inverse Matrix
This module introduces the inverse matrix and its properties.

We have seen that the number 1 plays a special role in multiplication, because $1x = x$.

The **inverse of a number** is defined as the number that multiplies by that number to give 1: b is the inverse of a if $ab = 1$. Hence, the inverse of 3 is $\frac{1}{3}$; the inverse of $\frac{-5}{8} = \frac{-8}{5}$. Every number except 0 has an inverse.

By analogy, the **inverse of a matrix** multiplies by that matrix to give the identity matrix.

Definition of Inverse Matrix
The **inverse of matrix** $[A]$, designated as $[A]^{-1}$, is defined by the property:
$[A][A]^{-1} = [A]^{-1}[A] = [I]$

The superscript –1 is being used here in a similar way to its use in functions. Recall that $f^{-1}(\mathrm{x})$ does not designate an exponent of any kind, but instead, an inverse function. In the same way, $[A]^{-1}$ does not denote an exponent, but an inverse **matrix**.

Note that, just as in the definition of the identity matrix, this definition requires commutativity—the multiplication must work the same in either order.

Note also that only square matrices can have an inverse. Why? The definition of an inverse matrix is **based on** the identity matrix $[I]$, and we already said that only square matrices even **have** an identity!

How do you **find** an inverse matrix? The method comes directly from the definition, with a little algebra.

Example: Finding an Inverse Matrix	
Find the inverse of $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$	**The problem**

Example: Finding an Inverse Matrix	
$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**This is the key step. It establishes** $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ **as the inverse that we are looking for, by asserting that it fills the definition of an inverse matrix: when you multiply this mystery matrix by our original matrix, you get [I]. When we solve for the four variables a, b, c, and d, we will have found our inverse matrix.**

Example: Finding an Inverse Matrix	
$\begin{bmatrix} 3a+4c & 3b+4d \\ 5a+6c & 5b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**Do the multiplication. (You should check this step for yourself, it’s great practice. For instance, you start by multiplying first row x first column, and you get 3a+4c.)**
$3a+4c=1 \quad 3b+4d=0 \quad 5a+6c=0 \quad 5b+6d=1$	**Remember what it means for two matrices to be equal: every element in the left must equal its corresponding element on the right. So, for these two matrices to equal each other, all four of these equations must hold.**

Example: Finding an Inverse Matrix	
$a = -3 \quad b = 2 \quad c = 2\frac{1}{2} \quad d = -1\frac{1}{2}$	**Solve the first two equations for a and c by using either elimination or substitution. Solve the second two equations for b and d by using either elimination or substitution. (The steps are not shown here.)**
So the inverse is: $\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix}$	**Having found the four variables, we have found the inverse.**

Did it work? Let's find out.

Testing our Inverse Matrix	
$\begin{bmatrix} -3 & 2 \\ 2\frac{1}{2} & -1\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**The definition of an inverse matrix: if we have indeed found an inverse, then when we multiply it by the original matrix, we should get [I].**

Testing our Inverse Matrix	
$\begin{bmatrix} (-3)(3)+(2)(5) & (-3)(4)+(2)(6) \\ \left(2\frac{1}{2}\right)(3)+\left(-1\frac{1}{2}\right)(5) & \left(2\frac{1}{2}\right)(4)+\left(-1\frac{1}{2}\right)(6) \end{bmatrix}$ $\stackrel{?}{=}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**Do the multiplication.**
$\begin{bmatrix} -9+10 & -12+12 \\ 7\frac{1}{2}-7\frac{1}{2} & 10-9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	**It works!**

Note that, to fully test it, we would have to try the multiplication in **both orders**. Why? Because, in general, changing the order of a matrix multiplication changes the answer; but the definition of an inverse matrix specifies that it must work both ways! Only one order was shown above, so technically, we have only half-tested this inverse.

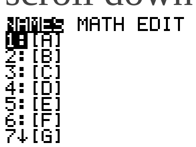
This process does not have to be memorized: it should make logical sense. Everything we have learned about matrices should make logical sense, except for the very arbitrary-looking definition of matrix multiplication.

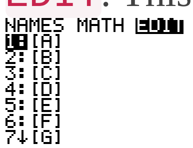
Matrix Concepts -- Matrices on Calculators

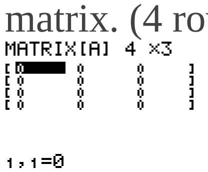
Many modern graphing calculators have all the basic matrix operations built into them. The following is a brief overview of how to work with matrices on a TI-83, TI-83 Plus, TI-84, or TI-84 Plus.

The calculator has room to store up to ten matrices at once. It refers to these matrices as [A], [B], and so on, through [J]. Note that these are **not** the same as the 26 lettered memories used for numbers.

The following steps will walk you through the process of entering and manipulating matrices.

1. Hit the `MATRX` button. On a TI-83, this is a standalone button; on a TI-83 Plus, you first hit `2nd` and then `MATRIX` (above the x^{-1} button). The resulting display is a list of all the available matrices. (You have to scroll down if you want to see the ones below `[G]`.)


2. Hit the **right arrow key** ► twice, to move the focus from `NAMES` to `EDIT`. This signals that you want to create, or change, a matrix.


3. Hit the number `1` to indicate that you want to edit the **first** matrix, `[A]`.
4. Hit `4 ENTER 3 ENTER` to indicate that you want to create a 4x3 matrix. (4 rows, 3 columns.)


5. Hit `1 ENTER 2 ENTER 3 ENTER 4 ENTER 5 ENTER 6 ENTER 7 ENTER 8 ENTER 9 ENTER 10 ENTER 11 ENTER 12 ENTER` This fills in the matrix with those numbers (you can watch it fill as you go). If you make a mistake, you can use the arrow keys to move around in the matrix until the screen looks like the picture below.

```
MATRIX[A] 4 ×3
[1    2    3    ]
[4    5    6    ]
[7    8    9    ]
[10   11   12   ]

4,3=12
```

6. Hit `2nd Quit` to return to the main screen.
7. Return to the main matrix menu, as before. However, this time, do **not** hit the right arrow to go to the `EDIT` menu. Instead, from the NAMES menu, hit the number `1`. This puts `[A]` on the main screen. Then hit `ENTER` to display matrix `[A]`.

```
[A]
     [[1  2  3 ]
      [4  5  6 ]
      [7  8  9 ]
      [10 11 12]]
```

8. Go through the process (steps 1-7) again, with a few changes. This time, define matrix `[B]` instead of matrix `[A]`. (This will change step 3: once you are in the `EDIT` menu, you will hit a `2` instead of a 1.) Define `[B]` as a 3x2 matrix in step 4. Then, in step 5, enter the following numbers:
   Equation:

$$\begin{matrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{matrix}$$

   When you are done, and have returned to the main screen and punched 2 in the NAMES menu (step 7), your main screen should look like this:

```
      [4  5  6 ]
      [7  8  9 ]
      [10 11 12]]
[B]
        [[10 40]
         [20 50]
         [30 60]]
```

9. Now, type the following keys, watching the calculator as you do so. TI-83 Plus users should always remember to hit `2nd MATRIX` instead of just `MATRX.` `MATRX 1 + MATRX 2`

```
      [4  5  6 ]
      [7  8  9 ]
      [10 11 12]]
[B]
        [[10 40]
         [20 50]
         [30 60]]
[A]+[B]
```

   This instructs the computer to add the two matrices. Now hit `ENTER`

```
ERR:DIM MISMATCH
1:Quit
2:Goto
```

Hey, what happened? You asked the computer to add two matrices. But these matrices have **different dimensions**. Remember that you can only add two matrices if they have the same dimensions—that is, the same number of rows as columns. So you got an “Error: Dimension Mismatch.” Hit `ENTER` to get out of this error and return to the main screen.

10. Now try the same sequence without the + key: `MATRX 1 MATRX 2 ENTER`

```
             [20 50]
             [30 60]]
[A][B]
       [[140 320 ]
        [320 770 ]
        [500 1220]
        [680 1670]]
```

This instructs the calculator to **multiply** the two matrices. This is a legal multiplication—in fact, you may recognize it as the multiplication that we did earlier. The calculator displays the result that we found by hand: $\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{matrix} \quad \begin{matrix} 10 & 40 \\ 20 & 50 \\ 30 & 60 \end{matrix} = \begin{matrix} 140 & 320 \\ 320 & 770 \\ 500 & 1220 \\ 680 & 1670 \end{matrix}$

11. Enter a third matrix, matrix [C]= $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. When you confirm that it is entered correctly, the screen should look like this:

```
[C]
              [[3 4]
               [5 6]]
```

Now type `MATRX 3 x-1 ENTER`

```
[C]
              [[3 4]
                [5 6]]
[C]⁻¹
       [[-3  2   ]
        [2.5 -1.5]]
```

This takes the **inverse** of matrix [C]. Note that the answer matches the inverse matrix that we found before.

12. Type `MATRX 3 x-1 MATRX 3 ENTER`

```
               [5 6]]
[C]⁻¹
      [[-3  2    ]
       [2.5 ⁻1.5]]
[C]⁻¹[C]
             [[1 0]
              [0 1]]
```

This instructs the calculator to multiply matrix [C]-1 times matrix [C]. The answer, of course, is the 2×2 identity matrix [I].

Matrix Concepts -- Determinants
This module covers matrix determinants and their uses.

The Determinant of a 2x2 Matrix

In the exercise "Inverse of the Generic 2x2 Matrix," you found that the inverse of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{\mathrm{ad-bc}}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. This formula can be used to very quickly find the inverse of any 2x2 matrix.

Note that if $ad – bc = 0$, the formula does not work, since it puts a 0 in the denominator. This tells us that, for any 2x2 matrix, if $ad – bc = 0$ the matrix **has no inverse**.

The quantity ad–bc is therefore seen to have a special importance for 2x2 matrices, and it is accorded a special name: the "determinant." Determinants are represented mathematically with absolute value signs: the determinant of matrix [A] is |A|.

Definition of the Determinant of a 2x2 Matrix
If matrix $\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the determinant is the number $|A| = ad - bc$.

For instance, for the matrix $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, the determinant is (3)(6)–(4)(5) = –2.

Note that the determinant is a **number**, not a matrix. It is a special number that is **associated with** a matrix.

We said earlier that "if $ad – bc = 0$ the matrix **has no inverse**." We can now restate this result.

Any square matrix whose determinant is **not** 0, has an inverse matrix. Any square matrix with determinant 0 has no inverse.

This very important result is analogous to the result stated earlier for numbers: every number except 0 has an inverse.

The Determinant of a 3x3 Matrix (or larger)

Any **square matrix** has a determinant—an important number associated with that matrix. Non-square matrices do not have a determinant.

How do you find the determinant of a 3x3 matrix? The method presented here is referred to as "expansion by minors." There are other methods, but they turn out to be mathematically equivalent to this one: that is, they end up doing the same arithmetic and arriving at the same answer.

Example: Finding the Determinant of a 3x3 Matrix	
Find the determinant of $\begin{matrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{matrix}$	**The problem.**
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED.	**We're going to walk through the top row, one element at a time, starting with the first element (the 2). In each case, begin by crossing out the row and column that contain that number.**
$\left\lvert \begin{matrix} 8 & 3 \\ 1 & 1 \end{matrix} \right\rvert = (8)(1) - (3)(1) = 5$	**Once you cross out one row and column, you are left with a 2x2 matrix (a "minor"). Take the determinant of that matrix.**
2(5)=10	**Now, that "minor" is what we got by crossing out a 2 in the top row. Multiply that number in the top row (2) by the determinant of the minor (5).**

Example: Finding the Determinant of a 3x3 Matrix	
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. (10)(1)–(3)(1)= 74(7)=28	**Same operation for the second element in the row (the 4 in this case)...**
SORRY, THIS MEDIA TYPE IS NOT SUPPORTED. (10)(1)–(8)(1) = 25(2)=10	**...and the third (the 5 in this case).**
+10 – 28 + 10 = –8	**Take these numbers, and alternately add and subtract them; add the first, subtract the second, add the third. The result of all that is the determinant.**

This entire process can be written more concisely as:

$$\begin{matrix} 2 & 4 & 5 \\ 10 & 8 & 3 \\ 1 & 1 & 1 \end{matrix} = 2 \begin{matrix} 8 & 3 \\ 1 & 1 \end{matrix} - 4 \begin{matrix} 10 & 3 \\ 1 & 1 \end{matrix} + 5 \begin{matrix} 10 & 8 \\ 1 & 1 \end{matrix} = 2 \quad 5 \quad - 4 \quad 7 \quad + 5 \quad 2 \quad = -8$$

This method of “expansion of minors” can be extended upward to any higher-order square matrix. For instance, for a 4x4 matrix, each “minor” that is left when you cross out a row and column is a 3x3 matrix. To find the determinant of the 4x4, you have to find the determinants of all four 3x3 minors!

Fortunately, your calculator can also find determinants. Enter the matrix given above as matrix [D]. Then type:

`MATRX` ▶ `1MATRX 4 ) ENTER`

The screen should now look like this:

```
[D]
      [[2  4 5]
       [10 8 3]
       [1  1 1]]
det([D])
                  -8
```

If you watched the calculator during that sequence, you saw that the right-arrow key took you to the `MATH` submenu within the `MATRIX` menus. The first item in that submenu is `DET (` which means "determinant of."

What does the determinant mean? It turns out that this particular odd set of operations has a surprising number of applications. We have already seen one—in the case of a 2x2 matrix, the determinant is part of the inverse. And for any square matrix, the determinant tells you whether the matrix **has** an inverse at all.

Another application is for finding the area of triangles. To find the area of a triangle whose vertices are (a,b), (c,d), and (e,f), you can use the formula: Area = ½ $\begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}$. Hence, if you draw a triangle with vertices (2,10), (4,8), and (5,3), the above calculation shows that the area of this triangle will be 4.

Matrix Concepts -- Solving Linear Equations
This module explains how to use matrices to solve linear equations.

At this point, you may be left with a pretty negative feeling about matrices. The initial few ideas—adding matrices, subtracting them, multiplying a matrix by a constant, and matrix equality—seem almost too obvious to be worth talking about. On the other hand, multiplying matrices and taking determinants seem to be strange, arbitrary sequences of steps with little or no purpose.

A great deal of it comes together in solving linear equations. We have seen, in the chapter on simultaneous equations, how to solve two equations with two unknowns. But suppose we have three equations with three unknowns? Or four, or five? Such situations are more common than you might suppose in the real world. And even if you are allowed to use a calculator, it is not at all obvious how to solve such a problem in a reasonable amount of time.

Surprisingly, the things we have learned about matrix multiplication, about the identity matrix, about inverse matrices, and about matrix equality, give us a very fast way to solve such problems on a calculator!

Consider the following example, three equations with three unknowns:
Equation:

$$x + 2y - z = 11$$

Equation:

$$2x - y + 3z = 7$$

Equation:

$$7x - 3y - 2z = 2$$

Define a 3×3 matrix [A] which is the coefficients of all the variables on the left side of the equal signs:

$$
[A] = \begin{matrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{matrix}
$$

Define a 3×1 matrix [B] which is the numbers on the **right** side of the equal signs:

$$
[B] = \begin{matrix} 11 \\ 7 \\ 2 \end{matrix}
$$

Punch these matrices into your calculator, and then ask the calculator for [A-1][B]: that is, the inverse of matrix [A], multiplied by matrix [B].

```
        [[11]
         [7 ]
         [2 ]]
[A]-1[B]
         [[3]
          [5]
          [2]]
```

The calculator responds with a 3×1 matrix which is **all three answers**. In this case, $x = 3$, $y = 5$, and $z = 2$.

The whole process takes no longer than it takes to punch a few matrices into the calculator. And it works just as quickly for 4 equations with 4 unknowns, or 5, **etc.**

Huh? Why the heck did that work?

Solving linear equations in this way is fast and easy. But with just a little work—and with the formalisms that we have developed so far about matrices—we can also show why this method works.

Step 1: In Which We Replace Three Linear Equations With One Matrix Equation

First of all, consider the following matrix equation:

$$\begin{matrix} x+2\mathrm{y}-z \\ 2\mathrm{x}-y+3\mathrm{z} \\ 7\mathrm{x}-3\mathrm{y}-2\mathrm{z} \end{matrix} = \begin{matrix} 11 \\ 7 \\ 2 \end{matrix}$$

The matrix on the left may **look** like a 3×3 matrix, but it is actually a 3×1 matrix. The top element is $x+2y-z$ (all one big number), and so on.

Remember what it means for two matrices to be equal to each other. They have to have the same dimensions (

). And all **the elements have to be equal to each other**. So for this matrix equation to be true, all three of the following equations must be satisfied:
Equation:

$$x+2y–z=11$$

Equation:

$$2x–y+3z=7$$

Equation:

$$7x–3y–2z=2$$

Look familiar? Hey, this is the three equations we started with! The point is that this **one matrix equation** is equivalent to those **three linear equations**. We can replace the original three equations with one matrix equation, and then set out to solve that.

Step 2: In Which We Replace a Simple Matrix Equation with a More Complicated One

Do the following matrix multiplication. (You will need to do this by hand—since it has variables, your calculator can't do it for you.)

$$\begin{matrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{matrix} \quad \begin{matrix} x \\ y \\ z \end{matrix}$$

If you did it correctly, you should have wound up with the following 3×1 matrix:

Equation:

$$\begin{matrix} x + 2\text{y} - z \\ 2\text{x} - y + 3\text{z} \\ 7\text{x} - 3\text{y} - 2\text{z} \end{matrix}$$

Once again, we pause to say…hey, that looks familiar! Yes, it's the matrix that we used in Step 1. So we can now rewrite the **matrix equation** from Step 1 in this way:

$$\begin{matrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{matrix} \quad \begin{matrix} x \\ y \\ z \end{matrix} = \begin{matrix} 11 \\ 7 \\ 2 \end{matrix}$$

Stop for a moment and make sure you're following all this. I have shown, in two separate steps, that this **matrix equation** is equivalent to the three **linear equations** that we started with.

But this matrix equation has a nice property that the previous one did not. The first matrix (which we called [A] a long time ago) and the third one ([B]) contain only numbers. If we refer to the middle matrix as [X] then we can write our equation more concisely:

$$[A][X] = [B]\text{, where } [A] = \begin{matrix} 1 & 2 & -1 \\ 2 & -1 & 3 \\ 7 & -3 & -2 \end{matrix}\text{, } [X] = \begin{matrix} x \\ y \\ z \end{matrix}\text{, and } [B] = \begin{matrix} 11 \\ 7 \\ 2 \end{matrix}$$

Most importantly, $[X]$ contains the three variables we want to solve for! If we can solve this equation for $[X]$ we will have found our three variables x,

y, and z.

Step 3: In Which We Solve a Matrix Equation

We have rewritten our original equations as $[A][X] = [B]$, and redefined our original goal as "solve this matrix equation for $[X]$." If these were numbers, we would divide both sides by $[A]$. But these are matrices, and we have never defined a division operation for matrices. Fortunately, we can do something just as good, which is **multiplying both sides** by $[A]^{-1}$. (Just as, with numbers, you can replace "dividing by 3" with "multiplying by $\frac{1}{3}$.")

Solving a Matrix Equation	
$[A][X] = [B]$	**The problem.**
$[A]^{-1}[A][X] = [A]^{-1}[B]$	**Multiply both sides by $[A]^{-1}$, on the left. (Remember order matters! If we multiplied by $[A]^{-1}$ on the right, that would be doing something different.)**
$[I][X] = [A]^{-1}[B]$	**$[A]^{-1}[A] = [I]$ by the definition of an inverse matrix.**
$[X] = [A]^{-1}[B]$	**$[I]$ times anything is itself, by definition of the identity matrix.**

So we're done! $[X]$, which contains exactly the variables we are looking for, has been shown to be $[A]^{-1}[B]$. This is why we can punch that formula into our calculator and find the answers instantly.

Let’s try one more example

Equation:

$$5x – 3y – 2z = 4$$

Equation:

$$x + y – 7z = 7$$

Equation:

$$10x – 6y – 4z = 10$$

We don’t have to derive the formula again—we can just use it. Enter the following into your calculator:

$$[A] = \begin{matrix} 5 & -3 & -2 \\ 1 & 1 & -7 \\ 10 & -6 & -4 \end{matrix} \quad [B] = \begin{matrix} 4 \\ 7 \\ 10 \end{matrix}$$

Then ask the calculator for $[A]^{-1}[B]$.

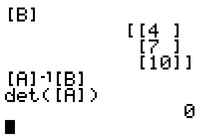
```
    [[5   -3  -2]
     [1   1   -7]
     [10  -6  -4]]
[B]
          [[4 ]
           [7 ]
           [10]]
[A]-1[B]
```

The result?

```
ERR:SINGULAR MAT
1:Quit
2:Goto
```

What happened? To understand this error, try the following:

Hit ENTER to get out of the error, and then hit <MATRX> ► 1 <MATRX> 1) ENTER



Aha! Matrix $[A]$ has a determinant of 0. **A matrix with 0 determinant has no inverse.** So the operation you asked the calculator for, $[A]^{-1}[B]$, is impossible.

What does this tell us about our original equations? They have no solution. To see why this is so, double the first equation and compare it with the third—it should become apparent that both equations cannot be true at the same time.